

Functorial relationship between multirings and the various abstract theories of quadratic forms

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October 5, 2016

Abstract

We register, explicitly, equivalences and dual equivalences between categories of abstract quadratic forms theories and (sub)categories of multirings and multifields.

1 Multigroups and Multirings

This section is a compiled o basic definitions and results included for the convenience of the reader. For more details, consult [6].

Multigroups are a generalization of groups. We can think that a multigroup is a group with a multivalued operation:

Definition 1.1 A multigroup is a quadruple $(G, *, r, 1)$, where G is a non-empty set,

$*$: $G \times G \rightarrow \mathcal{P}(G) \setminus \{\emptyset\}$ and $r : G \rightarrow G$ are functions, and 1 is an element of G satisfying:

i - If $z \in x * y$ then $x \in z * r(y)$ and $y \in r(x) * z$.

ii - $y \in 1 * x$ iff $x = y$.

iii - With the convention $x * (y * z) = \bigcup_{w \in y * z} x * w$ and $(x * y) * z = \bigcup_{t \in x * y} t * z$,

$$x * (y * z) = (x * y) * z \text{ for all } x, y, z \in G.$$

A multigroup is said to be *commutative* if

iv - $x * y = y * x$ for all $x, y \in G$.

Example 1.2

a - Suppose $(G, \cdot, 1)$ is a group. Defining $*(a, b) = \{c \in G : c = a \cdot b\}$ and $r(g) = g^{-1}$, we have that $(G, *, r, 1)$ is a multigroup.

We have too, an another description to multigroups, due by Marshall in [6]:

Definition 1.3 A multigroup is a quadruple (G, Π, r, i) where G is a non-empty set, Π is a subset of $G \times G \times G$, $r : G \rightarrow G$ is a function and i is an element of G satisfying:

I - If $(x, y, z) \in \Pi$ then $(z, r(y), x) \in \Pi$ and $(r(x), z, y) \in \Pi$.

II - $(x, i, y) \in \Pi$ iff $x = y$.

III - If $\exists p \in G$ such that $(u, v, p) \in \Pi$ and $(p, w, x) \in \Pi$ then $\exists q \in G$ such that $(v, w, q) \in \Pi$ and $(u, q, x) \in \Pi$.

A multigroup is said to be *commutative* if

IV - $(x, y, z) \in \Pi$ iff $(y, x, z) \in \Pi$.

In fact, these definitions describes the same object, and that connection is established by the following lemma:

Lemma 1.4 For any multigroup G as in the second version, we have:

a - $r(i) = i$.

b - $r(r(x)) = x$.

c - $(x, y, z) \in \Pi$ iff $(r(y), r(x), r(z)) \in \Pi$.

d - $(i, x, y) \in \Pi$ iff $x = y$.

e - If $\exists q \in G$ such that $(v, w, q) \in \Pi$ and $(u, q, x) \in \Pi$ then $\exists p \in G$ such that $(u, v, p) \in \Pi$ and $(p, w, x) \in \Pi$.

f - For each $a, b \in G$, there exists $c \in G$ such that $(a, b, c) \in \Pi$.

Proof:

a - As $i = i$, $(i, i, i) \in \Pi$ by II. By I, $(r(i), i, i) \in \Pi$ and by II, $r(i) = i$.

b - $x = x \xRightarrow{II} (x, i, x) \in \Pi \xRightarrow{I} (r(x), x, i) \in \Pi \xRightarrow{I} (r(r(x)), i, x) \in \Pi \xRightarrow{II} r(r(x)) = x$.

c - $(x, y, z) \xRightarrow{I} (z, r(y), x) \in \Pi \xRightarrow{I} (r(z), x, r(y)) \in \Pi \xRightarrow{I} (r(y), r(x), r(z)) \in \Pi$.

d - Let $(i, x, y) \in \Pi$.

$$\begin{aligned} (i, x, y) \in \Pi &\xRightarrow{I} (y, r(x), i) \in \Pi \xRightarrow{I} (r(y), i, r(x)) \in \Pi \\ &\xRightarrow{I} r(y) = r(x) \xRightarrow{(b)} y = r(r(y)) = r(r(x)) = x. \end{aligned}$$

Conversely, suppose $x = y$.

$$\begin{aligned} x = y &\Rightarrow r(x) = r(y) \xRightarrow{II} (r(y), i, r(x)) \in \Pi \\ &\xRightarrow{I+(b)} (y, r(x), i) \in \Pi \xRightarrow{I} (i, x, y) \in \Pi. \end{aligned}$$

e - $(u, q, x) \in \Pi \xRightarrow{I} (x, r(q), u) \in \Pi \xRightarrow{(c)} (q, r(x), r(u)) \in \Pi$. Then, $(v, w, q) \in \Pi$ and $(q, r(x), r(u)) \in \Pi$, so by axiom III, there exists $t \in G$ such that $(w, r(x), t) \in \Pi$ and $(v, t, r(u)) \in \Pi$.

$$(w, r(x), t) \in \Pi \xRightarrow{(b)} (x, r(w), t) \in \Pi \xRightarrow{I} (r(t), w, x) \in \Pi, \text{ and}$$

$$(v, t, r(u)) \in \Pi \xRightarrow{(b)} (r(t), r(v), u) \in \Pi \xRightarrow{I} (u, v, r(t)) \in \Pi.$$

Hence Defining $p = r(t)$, we have $(u, v, p) \in \Pi$ and $(p, w, x) \in \Pi$.

f - Hence $(b, r(b), i) \in \Pi$ and $(a, i, a) \in \Pi$, by (e), there exists $c \in G$ such that $(a, b, c) \in \Pi$ and $(c, r(b), a) \in \Pi$.

■

Now, let $(G, *, r, 1)$ a multigroup in the sense 1.1. We can define a multigroup (G, Π_*, r, i) taking $i = 1$ and $\Pi_* = \{(a, b, c) : c \in a * b\}$. The validate of the axioms I, II, III (and IV) for (G, Π_*, r, i) are direct consequence of axioms i, ii, iii and (iv) in $(G, *, r, 1)$.

Conversely, let (G, Π, r, i) a multigroup in the sense 1.3. By 1.4(f), the function $*_{\Pi} : A \times A \rightarrow \mathcal{P}(A) \setminus \{\emptyset\}$, gives by $*_{\Pi}(a, b) = a *_{\Pi} b := \{c \in G : (a, b, c) \in \Pi\}$ is well defined. Hence, Let $(G, *_{\Pi}, 1)$ with $1 = i$. Then, the validate of the axioms i, ii (and iv) for $(G, *_{\Pi}, 1)$ are direct consequence of I, II (and IV) for (G, Π, r, i) . For the axiom iii, let $x \in a *_{\Pi} (b *_{\Pi} c)$. Then $x \in a *_{\Pi} q$ for some $q \in b *_{\Pi} c$. As $(b, c, q) \in \Pi$ and $(a, q, x) \in \Pi$, by 1.4(e), there exists $p \in \Pi$ such that $(a, b, p) \in \Pi$ and $(p, c, x) \in \Pi$ and then, $x \in p *_{\Pi} c$ with $p \in a *_{\Pi} b$ that imply $x \in (a *_{\Pi} b) *_{\Pi} c$. Finally, let $y \in (a *_{\Pi} b) *_{\Pi} c$. So $y \in p *_{\Pi} c$ for some $p \in a *_{\Pi} b$, then and $(a, b, p) \in \Pi$ and $(p, c, y) \in \Pi$. By III, there exists $q \in \Pi$ such that $(b, c, q) \in \Pi$ and $(a, q, y) \in \Pi$. Hence $y \in a *_{\Pi} q$ and $q \in b *_{\Pi} c$, that imply $y \in a *_{\Pi} (b *_{\Pi} c)$. Therefore, $(G, *_{\Pi}, 1)$ is a multigroup in the sense 1.1.

From here, we will define multirings and study this structure with more details:

Definition 1.5 A multiring is a sextuple $(R, +, \cdot, -, 0, 1)$ where R is a non-empty set, $+, - : R \times R \rightarrow \mathcal{P}(R) \setminus \{\emptyset\}$, $\cdot : R \times R \rightarrow R$ and $- : R \rightarrow R$ are functions, 0 and 1 are elements of R satisfying:

- i - $(R, +, -, 0)$ is a commutative multigroup;
- ii - $(R, \cdot, 1)$ is a commutative monoid;
- iii - $a0 = 0$ for all $a \in R$;
- iv - If $c \in a + b$, then $cd \in ad + bd$. Or equivalently, $(a + b)d \subseteq ad + bd$.

R is said to be a multidomain if do not have zero divisors, and R will be a multifield if every non-zero element of R has multiplicative inverse. We will use two conventions: if $Z, W \subseteq R$ and $x \in R$, $Z + W = \bigcup\{x + y : x \in Z, y \in W\}$ and $Z + x = Z + \{x\} = \bigcup\{z + x : z \in Z\}$.

Example 1.6 a - As in example 1.2(a), every ring, domain and field is a multiring, multidomain and multifield respectively.

- b - $Q_2 = \{-1, 0, 1\}$ is a multifield with the usual product and the multivalued sum defined by relations

$$\begin{cases} 0 + x = x + 0 = x, \text{ for every } x \in Q_2 \\ 1 + 1 = 1, (-1) + (-1) = -1 \\ 1 + (-1) = (-1) + 1 = \{-1, 0, 1\} \end{cases}$$

- c - Let be $V \subseteq \mathbb{R}^n$ an algebraic set and A as the coordinate ring of V , i.e, the ring $\mathbb{R}[X]$ of polinomial functions $f : V \rightarrow \mathbb{R}$. Define an equivalence relation \sim on A by $f \sim g \Leftrightarrow f(x)$ and $g(x)$ has the same sign for all $x \in V$. Thus, $Q_{\text{red}}(A) = A / \sim$ is called the real reduced multiring. The operations are defined by:

$$\begin{cases} \overline{f} \in \overline{g} + \overline{h} \Leftrightarrow \exists f', g', h' \in A \\ \quad \text{such that } f' = g' + h', \overline{f'} = \overline{f}, \overline{g'} = \overline{g}, \text{ and } \overline{h'} = \overline{h} \\ \overline{gh} = \overline{g}\overline{h}, -\overline{f} = \overline{-f}, 0 = \overline{0}, 1 = \overline{1} \end{cases}$$

Taking $n = 1$, we have a counter-example to show that $ad + bd \subsetneq (a + b)d$ in general: $\frac{x^2 + x^3}{x^2 + x^3} \in \overline{xx} + \overline{x1}$ but $\frac{x^2 + x^3}{x^2 + x^3} \notin \overline{x(x + 1)}$, and this not happen because $x^2 + x^3 > 0$ and $x(x + 1) < 0$ for x near to 0 with $x \neq 0$.

- d - In the set \mathbb{R}_+ of positive real numbers, we define $a \nabla b = \{c \in \mathbb{R}_+ : |a - b| \leq c \leq a + b\}$. We have that \mathbb{R}_+ with the usual product and ∇ multivalued sum is a multifield, called triangle multifield [8]. We denote this multifield by $\mathcal{T}\mathbb{R}_+$.
- e - Let $K = \{0, 1\}$ with the usual product and the sum defined by relations $x + 0 = 0 + x = x$, $x \in K$ and $1 + 1 = \{0, 1\}$. This is a multifield called Krasner's multifield [5].

Lemma 1.7 *Let F be a multifield. Then $(a + b)d = ad + bd$ for every $a, b, d \in F$.*

Proof:

We have $(a + b)d \subseteq ad + bd$ already. For the other inclusion, if $d = 0$, it is done. If $d \neq 0$, we have:

$$\begin{aligned} (ad + bd)d^{-1} &\subseteq (ad)d^{-1} + (bd)d^{-1} = ad + bd \Rightarrow \\ ad + bd &= [(ad + bd)d^{-1}]d \subseteq (a + b)d. \end{aligned}$$

■

In the sequel, we will extend some terminology of commutative algebra from multirings and multifields. As expected, many concepts such that morphisms, ideals, fractions and localizations has a natural generalization for multirings. We treat of them and explain some pathologies that appears in the multivalued world.

Definition 1.8 An ideal of a multiring A is a non-empty subset of A such that $\mathfrak{a} + \mathfrak{a} \subseteq \mathfrak{a}$ and $A\mathfrak{a} = \mathfrak{a}$. An ideal \mathfrak{p} of A is said to be prime if $1 \notin \mathfrak{p}$ and $ab \in \mathfrak{p} \Rightarrow a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. An ideal \mathfrak{m} is maximal if $\mathfrak{m} \subseteq \mathfrak{a} \subseteq A \Rightarrow \mathfrak{a} = \mathfrak{m}$ or $\mathfrak{a} = A$. We will denote $\text{Spec}(A) = \{\mathfrak{p} \subseteq A : \mathfrak{p} \text{ is a prime ideal}\}$.

With the notion of ideal, we can define some new multirings structures with the language of commutative algebra in mind:

Definition 1.9

- a - If $\{A_i\}_{i \in I}$ is a family of multirings, then the product $\prod_{i \in I} A_i$ is a multiring in the natural (componentwise) way.
- b - Let $\mathfrak{a} \subseteq A$ an ideal. Elements of A/\mathfrak{a} are cosets $\bar{a} = a + \mathfrak{a}$, $a \in A$. We define a multiring structure on A/\mathfrak{a} by $\bar{a} + \bar{b} = \{\bar{c} : c \in a + b\}$, $-\bar{a} = \overline{-a}$, the zero and the unit element of A/\mathfrak{a} are $0 = \bar{0}$ and $1 = \bar{1}$ respectively and multiplication on A/\mathfrak{a} is defined by $\bar{a}\bar{b} = \overline{ab}$.
- c - Let S be a multiplicative set in A . Elements of $S^{-1}A$ have the form a/s , $a \in A$, $s \in S$, $a/s = b/t$ iff $atu = bsu$ for some $u \in S$. $0 = 0/1$, $1 = 1/1$ and the operations are defined by $(a/s) \cdot (b/t) = ab/st$, and $c/u \in a/s + b/t$ iff $cst \in atuv + bsuv$ for some $v \in S$.
- d - If D is a multidomain, we define the multifield of fractions $\text{ff}(D) := (D \setminus \{0\})^{-1}D$.

Now, we present a construction that will be used several times below:

Definition 1.10 Fix a multiring A and a multiplicative subset S of A . Define an equivalence relation \sim on A by $a \sim b$ iff $as = bt$ for some $s, t \in S$. Denote by \bar{a} the equivalence class of a and set $A/_m S = \{\bar{a} : a \in A\}$. Defining $\bar{a} + \bar{b} = \{\bar{c} : cv \in as + bt, \text{ for some } s, t, v \in S\}$, $-\bar{a} = \overline{-a}$, and $\bar{a}\bar{b} = \overline{ab}$ we have that $(A/_m S, +, \cdot, -, \bar{0}, \bar{1})$ is a multiring. When $S = \sum A^{*2}$, we will denote $A/_m \sum A^{*2} = Q_{\text{red}(A)}$.

Let S be a non-empty subset of a multiring A . We define the **ideal generated by S** as $\langle S \rangle := \bigcap \{\mathfrak{a} \subseteq A \text{ ideal} : S \subseteq \mathfrak{a}\}$. If $S = \{a_1, \dots, a_n\}$, we easily check that

$$\langle a_1, \dots, a_n \rangle = \sum Aa_1 + \dots + \sum Aa_n, \text{ where } \sum Aa = \bigcup_{n \geq 1} \underbrace{\{a + \dots + a\}}_{n \text{ times}}.$$

If A satisfy the second-half distributive, then $\sum Aa = Aa$. The following lemma is an useful property. The proof is the same of the ring case:

Lemma 1.11

- a - An ideal \mathfrak{p} of a multiring A is prime iff A/\mathfrak{p} is a multidomain.
- b - An ideal \mathfrak{m} of a multiring A is maximal iff A/\mathfrak{m} is a multifield.
- c - Every ideal maximal is prime.

Proposition 1.12 For any multiring A , $\text{Spec}(A)$ has a natural topology giving it the structure of a spectral space [4]. Basic open sets have the form $D(a) := \{\mathfrak{p} \in \text{Spec}(A) : a \notin \mathfrak{p}\}$.

Proof:

Let $\Phi : \text{Spec}(A) \rightarrow \{0, 1\}^A$ the embedding morphism given by

$$\Phi(\mathfrak{p}) = (x_a)_{a \in A}, \text{ where } x_a = \begin{cases} 0 & \text{if } a \in \mathfrak{p} \\ 1 & \text{if } a \notin \mathfrak{p} \end{cases}$$

The topology on $\text{Spec}(A)$ induced by Φ (giving $\{0, 1\}$ the discrete topology and $\{0, 1\}^A$ the product topology) is the so-called patch topology, i.e., the topology with subbasis consisting of the sets $\{D(a), \text{Spec}(A) \setminus D(b) : a, b \in A\}$. It suffices to show that $\text{Spec}(A)$ with the patch topology is a Boolean space or, equivalently, that the image of Φ is closed in $\{0, 1\}^A$ [4].

For this, we will decompose $\text{Im}(\Phi)$ as an intersection of closed subsets in $\{0, 1\}^A$. Observe that the $x = (x_a)_{a \in A} \in \text{Im}(\Phi)$ has the following relations:

$$\begin{cases} x_0 = 0, x_1 = 1 \\ x_a = 1 \text{ and } x_b = 1 \Rightarrow x_{ab} = 1 \\ x_a = 0 \text{ and } x_b = 0 \Rightarrow x_c = 1, \text{ for all } c \in a + b \end{cases} \quad (1)$$

For this reason, we define

$$\begin{cases} V_0 = \{0\} \times \prod_{a \in A \setminus \{0\}} \{0, 1\} \\ V_1 = \{1\} \times \prod_{a \in A \setminus \{1\}} \{0, 1\} \\ V_{a+b} = \{0\} \times \{0\} \times \prod_{c \in a+b} \{0\} \times \prod_{d \in A \setminus \{a+b \cup \{a, b\}\}} \{0, 1\} \\ V_{ab} = \left(\{0\} \times \{0\} \times \prod_{d \in A \setminus \{a, ab\}} \{0, 1\} \right) \cup \left(\{0\} \times \{0\} \times \prod_{d \in A \setminus \{b, ab\}} \{0, 1\} \right) \\ U_{ab} = \{1\} \times \{1\} \times \{1\} \times \prod_{d \in A \setminus \{a, b, ab\}} \{0, 1\} \\ B = \bigcup_{a, b \in A} V_{a+b}, C = \bigcup_{a, b \in A} V_{ab}, D = \bigcup_{a, b \in A} U_{ab} \end{cases}$$

We have that B, C and D are closed subsets, and by 1, $\text{Im}(\Phi) = V_0 \cap V_1 \cap B \cap C \cap D$. Indeed, if $x = \Phi(\mathfrak{p})$, the relations 1 asserts that $x \in V_0 \cap V_1 \cap B \cap C \cap D$, and if $y \in V_0 \cap V_1 \cap B \cap C \cap D$, the same relations implies that $\mathfrak{q} = \{a \in A : y_a = 0\}$ is a prime ideal of A . Then, $\text{Im}(\Phi)$ is closed in $\{0, 1\}^A$. ■

Now, we treat about morphisms:

Definition 1.13 Let A and B multirings. A map $f : A \rightarrow B$ is a morphism if for all $a, b, c \in A$:

i - $c \in a + b \Rightarrow f(c) \in f(a) + f(b)$;

ii - $f(-a) = -f(a)$;

iii - $f(0) = 0$;

iv - $f(ab) = f(a)f(b)$;

v - $f(1) = 1$.

For multirings, there are various sorts of “substructure” that one can consider. For rings, these all coincide. If A, B are multirings, we say A is embedded in B by the morphism $\iota : A \rightarrow B$ if ι is injective. We say A is strongly embedded in B if A is embedded in B and, for all $a, b, c \in A$, $\iota(c) \in \iota(a) +_B \iota(b) \Rightarrow c \in a +_A b$. We say A is a submultiring of B if A is strongly embedded in B and, for all $a, b \in A$ and all $c \in B$, $c \in \iota(a) +_B \iota(b) \Rightarrow c \in \iota(A)$.

The category of multifields (respectively multirings) and their morphisms will be denoted by \mathcal{MF} (respectively \mathcal{MR}). Some of the properties of rings morphisms are not extend to multirings morphisms. Next, are some counterexamples:

Example 1.14

- a - Let $f : \mathbb{R} \rightarrow Q_2$ be $f(x) = \text{sgn}(x)$, (with convention that $\text{sgn}(0) = 0$). f is a multiring morphism, but f is not injective and $\text{Ker} f = \{0\}$. Also $\mathbb{R}/\text{Ker} f$ is not isomorphic to Q_2 .
- b - The inclusions functions $Q_2 \hookrightarrow \mathbb{R}$ and $\mathcal{T}\mathbb{R}_+ \hookrightarrow \mathbb{R}$ are not multiring morphisms.

2 Ordering Structures

The standard Artin-Schreier can be extended to the multifield theory as in section 3 of [6]. For the convenience of the reader, we will list some results that we will use in the next sections:

Definition 2.1 Let F be a multifield. A subset P of F is called an *ordering* if $P + P \subseteq P$, $P \cdot P \subseteq P$, $P \cup -P = F$ and $P \cap -P = \{0\}$. The *real spectrum* of a multifield F , denoted $\text{Sper}(F)$, is defined to be the set of all orderings of F .

Definition 2.2 A *preordering* of a multifield F is defined to be a subset T of F satisfying $T + T \subseteq T$, $T \cdot T \subseteq T$ and $F^2 \subseteq T$. Here, $F^2 := \{a^2 : a \in F\}$. A multifield F is said to be *real* if $-1 \notin \sum F^2$. If F is real, then $-1 \neq 1$. A preordering T of F is said to be *proper* if $-1 \notin T$.

Proposition 2.3 Let F be an multifield and T a proper preordering of F . Then $T = \bigcap_{P \in X_T} P$, where $X_T = \{P \in \text{Sper}(F) : T \subseteq P\}$.

Proof:

Proposition 3.4 of [6]. ■

Consider the multifield Q_2 . $\{0, 1\}$ is an ordering on Q_2 . For any ordering P on a multifield F , $Q_P(F) = F/mP \cong Q_2$ by a unique isomorphism. Orderings of a multifield F correspond bijectively to a multiring homomorphism $\sigma : F \rightarrow Q_2$ via $P = \sigma^{-1}(\{0, 1\})$.

Proposition 2.4 For a real multifield F are equivalent:

- a - The multiring morphism $F \rightarrow Q_{\text{red}}(F)$ is an isomorphism;
- b - $\sum F^2 = \{0, 1\}$;

c - For all $a \in F$, $a^3 = a$ and $(a \in 1 + 1) \Rightarrow (a = 1)$.

Proof:

Proposition 4.1 [6]. ■

Definition 2.5 (Corollary 4.2 in [6]) A multifield F is said to be *real reduced* if $a^3 = a$ and $(a \in 1 + 1) \Rightarrow (a = 1)$ for all $a \in F$. This implies that the morphism $a \mapsto \bar{a}$ from F to $Q_{\text{red}}(F)$ is an isomorphism.

A morfism of real reduced multifields is just a morfism of multifields. The category of real reduced multifields will be denoted by $\mathcal{MF}_{\text{red}}$.

Theorem 2.6 (Local-Global principle) For any real reduced multifield F , the natural embedding $F \hookrightarrow Q_2^{\text{Sper}(F)}$ is a strong embedding.

Proof:

Proposition 4.4 [6]. ■

Definition 2.7 Let A be a multiring. A subset P of A is called an *ordering* if $P + P \subseteq P$, $P \cdot P \subseteq P$, $P \cup -P = A$ and $P \cap -P$ is a prime ideal of A , called the *support* of P . Orderings of a multiring A correspond bijectively to multiring homomorphisms $\sigma : A \rightarrow Q_2$ via $P = \sigma^{-1}(\{0, 1\})$. The *real spectrum* of a multiring A , denoted $\text{Sper}(A)$, is defined to be the set of all orderings of A .

Proposition 2.8 (Local-Global principle) Let A be a multiring with $-1 \notin \sum A^2$ and T a proper preordering of A . Then:

1. $Q_T(A)$ is a multiring.
2. $Q_T(A)$ is strong embedded in $Q_2^{X_T}$.

Proof:

Proposition 7.3 [6]. ■

Definition 2.9 (Corollary 7.6 in [6]) A multiring A is said to be *real reduced* if the following properties holds for all $a, b, c, d \in F$:

- i - $1 \neq 0$;
- ii - $a^3 = a$;
- iii - $c \in a + ab^2 \Rightarrow c = a$;
- iv - $c \in a^2 + b^2$ and $d \in a^2 + b^2$ implies $c = d$.

This implies that the morphism $a \mapsto \bar{a}$ from A to $Q_{\text{red}}(A)$ is an isomorphism.

A morfism of real reduced multirings is just a morfism of multirings. The category of real reduced multirings will be denoted by $\mathcal{MR}_{\text{red}}$.

3 Real Reduced Multifields and Abstract Ordering Spaces

In this section, we will construct a functor $\mathcal{MF}_{red} \rightarrow \mathcal{AOS}$ where \mathcal{AOS} is the category of Abstract Ordering Spaces as in [7]. This functor was cited by Marshall in [6], so we will make the details and examine some properties.

Recall that for any set X , $\{-1, 1\}^X$ is made into a group by defining $(ab)(x) = a(x)b(x)$, and that if G is a group of exponent 2, the *character group* of G is the group $\chi(G) := \text{Hom}(G, \mathbb{Z}_2)$ (here, we consider G and \mathbb{Z}_2 as topological groups).

Definition 3.1 (Space of Orderings) An *abstract ordering space* or *space of orderings*, abbreviated AOS, is a pair (X, G) satisfying:

AX1 - X is a non-empty set, G is a subgroup of $\{-1, 1\}^X$, G contains the constant function -1 , and G separates points in X (i.e, if $x, y \in X$, $x \neq y$, then there exists $a \in G$ such that $a(x) \neq a(y)$).

If $a, b \in G$ we define the **value set** $D(a, b)$ to be the set of all $c \in G$ such that for each $x \in X$ either $c(x) = a(x)$ or $c(x) = b(x)$. In particular, a and b are both elements of $D(a, b)$.

AX2 - If $x \in \chi(G)$ satisfies $x(-1) = -1$ and $a, b \in \ker(x) \Rightarrow D(a, b) \subseteq \ker(x)$, then x is in the image of the natural embedding $X \hookrightarrow \chi(G)$.

AX3 (Associativity) - For all $a, b, c \in G$, if $t \in D(a, r)$ for some $r \in D(b, c)$ then $t \in D(s, c)$ for some $s \in D(a, b)$.

Definition 3.2 A *morphism* α from an AOS (X, G) to an AOS (Y, H) is a mapping $\alpha : X \rightarrow Y$ such that for each $h \in H$, the composite function $h \circ \alpha : X \rightarrow \{-1, 1\}$ is an element of G (and in particular, α is surjective). Note that this implies that α induces a group homomorphism $h \mapsto h \circ \alpha$ from H to G . Also $\alpha^{-1}(U(h)) = U(h \circ \alpha)$ for each $h \in H$, so α is continuous.

An isomorphism from (X, G) to (Y, H) is a morphism $\alpha : X \rightarrow Y$ which is bijective and such that the induced group homomorphism $h \mapsto h \circ \alpha$ is also bijective.

Theorem 3.3 Let (X, G) a space of orderings and set $M(G) = G \cup \{0\}$ where $0 := \{G\}$. Then $(M(G), +, \cdot, -, 0, 1)$ is a real reduced multifield with the extended operations:

$$\begin{aligned} \bullet \quad a \cdot b &= \begin{cases} 0 & \text{if } a = 0 \text{ or } b = 0 \\ a \cdot b & \text{otherwise} \end{cases} \\ \bullet \quad -(a) &= (-1) \cdot a \\ \bullet \quad a + b &= \begin{cases} \{b\} & \text{if } a = 0 \\ \{a\} & \text{if } b = 0 \\ M(G) & \text{if } a = -b, \text{ and } a \neq 0 \\ D(a, b) & \text{otherwise} \end{cases} \end{aligned}$$

Proof:

Firstly, observe that $+$ is well-defined. Then, we will verify the conditions of definition 1.5:

i - For this, we will check the conditions of definition 1.1.

- a - If $a = 0$ or $a = -b$, then $d \in a + b$ implies trivially that $a \in d + (-b)$ and $b \in (-a) + d$. Now, let $a, b \neq 0$ with $a \neq -b$ (this implies $d \neq 0$). We prove that $(d(x) \in \{a(x), b(x)\} \forall x \in X) \Rightarrow (a(x) \in \{d(x), -b(x)\} \forall x \in X)$, and it is suffice for prove that $a \in d + (-b)$ and $b \in (-a) + d$. Let $x \in X$. If $c(x) = a(x)$ is done. If $c(x) \neq a(x)$ then $c(x) = b(x)$. If $c(x) = b(x) = 1$, then $a(x) = -1 = -b(x)$, and if $c(x) = b(x) = -1$, then $a(x) = 1 = -b(x)$, finalizing the argument.
- b - $(y \in x + 0) \Leftrightarrow (x = y)$ is direct consequence of the definition of sum.
- c - $a + 0 = 0 + a$ and $a + (-a) = M(G) = (-a) + a$. Let $a, b \in M(G)$, $a, b \neq 0$ and $a \neq -b$. How $D(a, b) = D(b, a)$, we have $a + b = b + a$. Then, the commutativity holds.
- d - Now we prove the associativity. Let $a = 0$ (the cases $b = 0$ and $c = 0$ are analogous). Then $0 + (b + c) = \{0 + g : g \in b + c\} = b + c$ and $(0 + b) + c = (\{b\}) + c = b + c$.
Now, let $a, b, c \neq 0$ with $a = -c$.

$$(a + b) + (-a) = \bigcup \{g + (-a) : g \in a + b\} = M(G) \text{ (I)}$$

because $a \in a + b$; and

$$a + (b + (-a)) = \bigcup \{a + h : h \in b + (-a)\} = M(G) \text{ (II)}$$

because $-a \in b + (-a)$. So (I) = (II) and $(a + b) + (-a) = a + (b + (-a))$. For the case $a, b, c \neq 0$, $a = -b$ (the cases $b \neq -c$ is analogous) we have

$$(a + (-a)) + c = \bigcup \{g + c : g \in M(G)\} = M(G) \text{ (III)}$$

and

$$a + ((-a) + c) = \bigcup \{a + h : h \in (-a) + c\} = M(G) \text{ (IV)}$$

because $-a \in (-a) + c$. So (III) = (IV) and $(a + (-a)) + c = a + ((-a) + c)$. Finally, let $a, b, c \neq 0$, $a \neq -b$, $b \neq -c$ and $a \neq -c$.

$$(a + b) + c = c + (a + b) = \bigcup \{c + g : g \in a + b\} = \bigcup_{g \in D(a, b)} D(c, g) \text{ (V)}$$

and

$$a + (b + c) = \bigcup \{h + a : h \in b + c\} = \bigcup_{h \in D(b, c)} D(h, a) \text{ (VI)}$$

By the inductive description of the value sets (as in 2.2 of [7]) we have (V) = (VI). Then $(a + b) + c = a + (b + c)$ for all $a, b, c \in M(G)$.

- ii - We conclude that $(M(G), \cdot, 1)$ is a commutative monoid as consequence of $(G, \cdot, 1)$ is an abelian group and the extended definition of \cdot to $M(G)$. Beyond this, we have that every nonzero element of $M(G)$ has an inverse.
- iii - $a \cdot 0 = 0$ for all $a \in M(G)$ is a consequence of the extended definition of multiplication to $M(G)$.
- iv - If $a = 0$ or $a \neq -b$, then $(d \in a + b) \Rightarrow \forall g(gd \in ga + gb)$ is direct consequence of the definition of sum. Next this, let $a, b \neq 0$ with $a \neq -b$ and $d \in a + b = D(a, b)$. Then $d(x) = a(x)$ or $d(x) = b(x)$ for all $x \in X$. Hence, $g(x)d(x) = g(x)a(x)$ or $g(x)d(x) = g(x)b(x)$ for all $x \in X$ and $gd \in ga + gb$. Thus we have $g(a + b) \subseteq ga + gb$ for all $a, b, g \in M(G)$.

Then, $(M(G), +, -, \cdot, 0, 1)$ is a multifield. As G is a subgroup of $\{-1, 1\}^X$, we have that G is a group of exponent 2, i.e, $g^2 = 1$ for all $g \in G$ and then, $a^3 = a$ for all $a \in M(G)$. If $a \in 1 + 1$, then $a(x) = x$ for all $x \in X$. This implies $a = 1$. Consequently, $M(G), +, -, \cdot, 0, 1$ is a real reduced multifield. ■

Corollary 3.4 *The correspondence $G \mapsto M(G)$ defines a contravariant functor $M : \mathcal{AOR}^{op} \rightarrow \mathcal{MF}_{red}$.*

Proof:

Let (X, G) and (Y, H) abstract ordering spaces and $\alpha : Y \rightarrow X$ be an AOR-morphism. By definition 3.2, α induces a group homomorphism $\varphi : G \rightarrow H$ given by $\varphi(g) = g \circ \alpha$. Define $M(\alpha) = \tilde{\varphi} : M(G) \rightarrow M(H)$ extending this morphism φ to $M(G)$ making $\tilde{\varphi}(0) = 0$. Note that we already have $\varphi(1) = 1$ and $\varphi(-1) = -1$.

Then, we just need to prove that for all $a, b, c \in G$, $c \in a + b \Rightarrow \tilde{\varphi} \in \tilde{\varphi} + \tilde{\varphi}$. We can suppose $a, b, c \neq 0$ and $a \neq -b$ without loss of generality. Hence, we will prove that $c \in D(a, b) \Rightarrow \varphi(c) \in D(\varphi(a), \varphi(b))$.

$$\begin{aligned} c \in D(a, b) &\Rightarrow c(x) = a(x) \vee c(x) = b(x) \forall x \in X \Rightarrow \\ c(\alpha(y)) &= a(\alpha(y)) \vee c(\alpha(y)) = b(\alpha(y)) \forall y \in Y \Rightarrow \\ c \circ \alpha &\in D(a \circ \alpha, b \circ \alpha) \Rightarrow D(\varphi(a), \varphi(b)) \end{aligned}$$

therefore $M(\varphi)$ is a MF-morphism. If $(Z, K) \xrightarrow{\beta} (Y, H) \xrightarrow{\alpha} (X, G)$ are AOR-morphism, with $\varphi : G \rightarrow H$ and $\tau : H \rightarrow K$ the respectively induced group homomorphisms, the fact of $M(\alpha\beta) = M(\beta)M(\alpha)$ is direct consequence of $\alpha\beta$ be an AOR-morphism. ■

Let F be an real reduced multifield. Observe that by the **local-global principle for multifield** we have the following identities:

- $a \in a + b$;
- If $a \neq 0$, then $a + (-a) = F$;
- $a \neq 0 \Rightarrow \sigma(a) \neq 0$ for all $\sigma \in \text{Sper}(F)$.

Now, let $\chi(F) = \{\sigma \in \{-1, 1\}^F : \sigma(ab) = \sigma(a)\sigma(b)\}$ and define

$$\begin{cases} X = \{x \in \chi(F) : x(-1) = -1 \text{ and } a, b \in \text{Ker}(x) \Rightarrow a + b \subseteq \text{Ker}(x)\} \\ G = \{\sigma \in \{-1, 1\}^X : \exists f \in F \text{ such that } \sigma(x) = x(f), \forall x \in X\} \end{cases}$$

Lemma 3.5 *There is a bijective correspondence between $(X, \text{Sper}(F))$ and (G, F) .*

Proof:

We will proof that the correspondence $x \mapsto x' : F \rightarrow Q_2$, $x'(f) = x(f)$ if $f \neq 0$ and $x'(0) = 1$ define a bijection $A : X \rightarrow \text{Sper}(F)$ and the correspondence $\sigma \mapsto f_\sigma$ when $\sigma(x) = x(f_\sigma)$ for all $x \in X$ define a bijection $B : G \rightarrow F$.

- A and B are well-defined. We need to prove that $x' : F \rightarrow Q_2$ is a multifield morphism and that $\bigcap_{x \in X} \text{Ker}(x) = \{1\}$, hence by this, $x(f) = x(g)$ for all $x \in X$ implies that $f g^{-1} \in \bigcap \text{Ker}(x) = \{1\}$ and then, $f = g$.

i - x' is a morphism. In fact, we just need to prove that $a \in b + c \Rightarrow x(a) \in x(b) + x(c)$. How the zero case is undefined, let $a, b, c \neq 0$. If $x(b) \neq x(c)$, then $x(b) + x(c) = Q_2$ and it is done. If $x(b) = x(c) = 1$, $a \in (b + c)^* \subseteq \text{Ker}(x) \Rightarrow x(a) \in x(b) + x(c)$. If $x(b) = x(c) = -1$, then $-a \in (-b - c)^* \subseteq \text{Ker}(x) \Rightarrow x(a) \in x(b) + x(c)$.

ii - $\bigcap_{x \in X} \text{Ker}(x) = \{1\}$. Let $a \neq 1$ in F^* . How F is a real reduced multifield,

$$a \notin \{0, 1\} = \sum F^2 = \bigcap_{P \in \text{Sper}(F)} P.$$

Let P an ordering such that $a \notin P$ and $\sigma : F \rightarrow Q_2$ its associate morfism. Note that $\sigma(a) = -1$ and $\sigma|_{F^*} \in X$, because

$$a, b \in \text{Ker}(\sigma) \Rightarrow \sigma(a + b) \subseteq \sigma(a) + \sigma(b) = \{1\} \Rightarrow (a + b)^* \subseteq \text{Ker}(\sigma|_{F^*})$$

Therefore $a \notin \bigcap_{x \in X} \text{Ker}(x)$.

- A and B are injective.

$$x \neq y \in X \Rightarrow \exists f \in F^* \text{ such that } x(f) \neq y(f) \Rightarrow x'(f) = x(f) \neq y(f) = y'(f).$$

$$\sigma \neq \gamma \in G \Rightarrow \exists x \in X \text{ such that } \sigma(x) \neq \gamma(x) \Rightarrow x(f_\sigma) \neq x(f_\gamma) \Rightarrow f_\sigma \neq f_\gamma.$$

- A and B are surjective. Given $\sigma \in \text{Sper}(F)$, we already proof that $\sigma|_{F^*} \in X$ and so $A(\sigma|_{F^*}) = \sigma$. For B , let $f \in F^*$, define $\sigma_f \in \{-1, 1\}^X$ given by $\sigma_f(x) = x(f)$ for all $x \in X$. Then $\sigma_f \in G$ and $B(\sigma_f) = f$.

■

Theorem 3.6 *With the above notation, (X, G) is an abstract ordering space.*

Proof:

Notation: if $\sigma \in G$, $f_\sigma = B(\sigma)$ and if $f \in F^*$, $\sigma_f = B^{-1}(f)$. Given $\sigma, \gamma \in G$, define $D(\sigma, \gamma) = \{\tau \in G : \forall x \in X, \tau(x) \in \{\sigma(x), \gamma(x)\}\}$. We have $D(\sigma, \gamma) = \{\tau : f_\tau \in (f_\sigma + f_\gamma)^*\}$.

$$\tau \in D(\sigma, \gamma) \Leftrightarrow \tau(x) = \sigma(x) \vee \tau(x) = \gamma(x) \Leftrightarrow \forall x \in X, \tau(x) \in \sigma(x) + \gamma(x)$$

$$\stackrel{\text{lemma 3.5}}{\Leftrightarrow} K \in \text{Sper}(F), K(f_\tau) \in K(f_\sigma) + (f_\gamma) \stackrel{\text{local-global principle}}{\Leftrightarrow} f_\tau \in (f_\sigma + f_\gamma).$$

Now, we will check each axiom of definition 3.1:

AX1 - $G \subseteq \{-1, 1\}^X$ is a subgroup, because $\sigma_f \sigma_g = \sigma_{fg}$, $1 \in G$ and $(\sigma_f)^{-1} = \sigma_{f^{-1}}$. Moreover, $-1 = \sigma_{-1} \in G$, because $x(-1) = -1$ for all $x \in X$. We already have that G separates points.

AX2 - Let $\Pi \in \chi(G)$ with $\Pi(\sigma_{-1}) = -1$ and $\sigma, \gamma \in \text{Ker}(\Pi) \Rightarrow D(\sigma, \gamma) \subseteq \text{Ker}(\Pi)$. We need to find $x \in X$ such that $\Pi(\sigma) = \sigma(x)$ for all $\sigma \in G$.

Define $x : F^* \rightarrow \{-1, 1\}$ by $x(f) = \Pi(\sigma_f)$. Note that $x \in \chi(F)$ and $x(-1) = -1$. To prove that $x \in X$ we need that $a, b \in \text{Ker}(x) \Rightarrow (a+b)^* \subseteq \text{Ker}(x)$.

$$a, b \in \text{Ker}(x) \Rightarrow \sigma_a, \sigma_b \in \text{Ker}(\Pi) \Rightarrow D(\sigma_a, \sigma_b) \subseteq \text{Ker}(\Pi)$$

Then

$$c \in (a+b)^* \Rightarrow \sigma_c \in D(\sigma_a, \sigma_b) \subseteq \text{Ker}(\Pi) \Rightarrow c \in \text{Ker}(x)$$

Therefore, $x \in X$. Moreover, given $\sigma = \sigma_f \in G$, we have

$$\Pi(\sigma) = \Pi(\sigma_f) = x(f) = \sigma_f(x) = \sigma(x)$$

finalizing the argument for AX2.

AX3 - Given $\sigma, \gamma, \tau \in G$, let $i \in D(\sigma, j)$ with $j \in D(\gamma, \tau)$. We will show that $i \in D(\sigma, j)$, $j \in D(\gamma, \tau) \Rightarrow f_i \in (f_\sigma + f_\tau)^*$ and $f_j \in (\gamma + f_\tau)^*$. How the sum in F is associative, there exist $l \in f_\sigma + f_\gamma$ with $f_i \in f_l + f_\gamma$.

If $l = 0$, we have $f_\sigma = -f_\gamma$ and $f_i = f_\gamma$ and then, $f_i \in (1 + f_\gamma)^*$ and $1 \in (f_\sigma + f_\gamma)^* \Rightarrow i \in D(\sigma_1, \gamma)$ and $\sigma_1 \in D(\sigma, \gamma)$. If $l \neq 0$, $i \in D(\sigma_l, \gamma)$ with $\sigma_l \in D(\sigma, \gamma)$.

■

Theorem 3.7 *There exist an equivalence of categories between \mathcal{AOR}^{op} and \mathcal{MF}_{red} .*

Proof:

Define $M : \mathcal{AOS}^{op} \rightarrow \mathcal{MF}_{Red}$ and $\text{Spec} : \mathcal{MF}_{Red} \rightarrow \mathcal{AOS}^{op}$ as we already defined in corollary 3.4 and theorem 3.6. Follow by that $M \circ \text{Spec} \cong \text{Id}_{\mathcal{MF}_{Red}}$ and $\text{Spec} \circ M \cong \text{Id}_{\mathcal{AOS}^{op}}$.

■

4 Real Reduced Multirings and Abstract Real Spectra

This section is dedicated to construct a functor $\mathcal{MR}_{red} \rightarrow \mathcal{ARS}$ where \mathcal{ARS} is the category of Abstract Real Spectra as in [7]. This is another functor cited by Marshall in [6] that we will analyze.

Recall that, $\{-1, 0, 1\}$ has a natural ordering relation and for any set X , $\{-1, 0, 1\}^X$ denotes the set of all functions $a : X \rightarrow \{-1, 0, 1\}$. This is a monoid with the operation given by $(ab)(x) = a(x)b(x)$.

Definition 4.1 (Abstract Real Spectra) An *abstract real spectrum or space of signs*, abbreviated to \mathcal{ARS} , is a pair (X, G) satisfying:

AX1 - X is a non-empty set, G is a submonoid of $\{-1, 0, 1\}^X$, G contains the constants functions $-1, 0, 1$, and G separates points in X .

If $a, b \in G$, the *value set* $D(a, b)$ is defined to be the set of all $c \in G$ such that, for all $x \in X$, either $a(x)c(x) > 0$ or $b(x)c(x) > 0$ or $c(x) = 0$. The *value set* $D^t(a, b)$ is defined to be the set of all $c \in G$ such that, for all $x \in X$, either $a(x)c(x) > 0$ or $b(x)c(x) > 0$ or $c(x) = 0$ and $b(x) = -a(x)$. Note that $c \in D^t(a, b) \Rightarrow c \in D(a, b)$. Conversely, $c \in D(a, b) \Rightarrow c \in D^t(ac^2, bc^2)$.

AX2 - If P is a submonoid of G satisfying $P \cup -P = G$, $-1 \notin P$, $a, b \in P \Rightarrow D(a, b) \subseteq P$ and $ab \in P \cap -P \Rightarrow a \in P \cap -P$ or $b \in P \cap -P$, then there exists $x \in X$ (necessarily unique) such that $P = \{a \in G : a(x) \leq 0\}$.

AX3 (Strong Associativity) - For all $a, b, c \in G$, if $p \in D^t(a, r)$ for some $q \in D^t(b, c)$ then $p \in D^t(r, c)$ for some $r \in D^t(a, b)$.

Definition 4.2 A *morphism* of ARS's $(X, G) \rightarrow (Y, H)$ is a mapping $\tau : X \rightarrow Y$ such that for each $a \in H$, the composite mapping is $a \circ \tau : X \rightarrow \{-1, 0, 1\}$ is an element of G (so τ is surjective and induces a mapping $a \mapsto a \circ \tau$ from H to G). τ is said to be an *isomorphism* if the mappings $X \rightarrow Y$ and $H \rightarrow G$ are bijective.

Theorem 4.3 Let (X, G) an abstract real spectra and define $a + b = \{d \in G : d \in D^t(a, b)\}$. Then $(G, +, \cdot, -, 0, 1)$ is a real reduced multiring.

Proof:

Firstly, observe that $+$ is well-defined. Then, we will verify the conditions of definition 1.5. Commutativity, associativity and neutral element ($a \in D^t(0, b) \Leftrightarrow a = b$) are immediate. In fact, the unique non-trivial part of the proof is

$$a \in D^t(b, c) \Rightarrow b \in D^t(a, -c) \text{ and } c \in D^t(-b, a).$$

We will prove that $b \in D^t(a, -c)$ and the case $c \in D^t(-b, a)$ analogous. Let $x \in X$ and $a \in D^t(b, c)$. Remember that $a \in D^t(b, c)$ means that $a(x)b(x) > 0$ or $a(x)c(x) > 0$ or $a(x) = 0$ and $b(x) = c(x)$ happens for all $x \in X$.

If $a(x)b(x) > 0$, then $b(x)a(x) > 0$ and it is done. If $a(x)c(x) > 0$, we have some cases:

- $a(x) = c(x) = 1$. We can suppose that $a(x)b(x) \leq 0$ and $b(x) \in \{0, 1\}$. If $b(x) = 0$ it is done. If $b(x) = 1$, then $b(x)[-c(x)] > 0$.
- $a(x) = c(x) = -1$. Again, we will suppose that $a(x)b(x) \leq 0$ and $b(x) \in \{0, 1\}$. If $b(x) = 0$ it is done. If $b(x) = 1$, then $b(x)[-c(x)] > 0$.
- $a(x) = 0$ and $b(x) = c(x)$. If $b(x) = c(x) = 0$ then $b(x) = 0$ and $a(x) = c(x)$. If $b(x) = c(x) \neq 0$, then $b(x)c(x) > 0$.

Hence G is a multiring. For the real reduced part, we have immediately that $1 \neq 0$ and $a^3 = a$ for all $a \in G$.

$$c \in D^t(a, ab^2) \Leftrightarrow c(x)a(x) = 0 \vee (c(x) = 0 \wedge a(x) = 0) \Leftrightarrow c = a$$

and

$$c \in D^t(a^2, b^2) \Leftrightarrow \forall x \in G(c(x) = 1 \vee (c(x) = 0 \wedge a(x)b(x) = 0))$$

This implies that c is uniquely determined. Therefore, G is a real reduced multiring. ■

Corollary 4.4 *There is a functor $M : \mathcal{ARS}^{op} \rightarrow \mathcal{MR}_{red}$.*

Proof:

Let (X, G) and (Y, H) be abstract real spectras and $\tau : Y \rightarrow X$ be a ARS-morphism. Define $M(X)$ how the real reduced multiring as in theorem 4.3 and $M(\tau) = f$ when $f : G \rightarrow H$ is the group homomorfism induced by τ . We have that $c \in a + b \Rightarrow c \in D^t(a, b) \Rightarrow f(c) \in D^t(f(a), f(b)) \Rightarrow f(c) \in f(a) + f(b)$ by an argument analogous to the corollary 6.8. Then $M(\tau)$ is a multiring morphism and this is suffice to prove that M is a (contravariant) functor. ■

Theorem 4.5 *Let A be an real reduced multiring and consider the strong embedding $i : A \rightarrow Q_2^{Sper(A)}$ given by $i(a) = \hat{a} : Sper(A) \rightarrow Q_2$ when $\hat{a}(\sigma) = \sigma(a)$. Define $\hat{A} = i(A)$. Then $(Sper(A), \hat{A})$ is an abstract real spectra.*

Proof:

We will check each definition of 4.1:

AX1 - Is consequence of \hat{A} be a submultiring of $Q_2^{Sper(A)}$.

AX2 - Let P be a submonoid of \hat{A} such that $P \cup -P = \hat{A}$, $-1 \notin P$, $a, b \in P \Rightarrow D(a, b) \subseteq P$ and $ab \in P \cap -P \Rightarrow a \in P \cap -P$ or $b \in P \cap -P$. First, For a Temer. Second, observe that

$$D^t(a, b) = \{d : d \in a + b\}. \quad (2)$$

In fact, $d \in D^t(a, b)$ if and only if $\forall \sigma \in Sper(A)$, $\sigma(d)\sigma(a) > 0$ or $\sigma(d)\sigma(b) > 0$ or $\sigma(d) = 0$, and $\sigma(a) = -\sigma(b)$ if and only if $\sigma(d) \in \sigma(a) + \sigma(b)$ for all $\sigma \in Sper(A)$. By the **local-global principle for multirings** we have that this happens if and only if $d \in a + b$.

AX3 - This is consequence of 2 and associativity. ■

Theorem 4.6 *There exist an equivalence of categories between \mathcal{ARS}^{op} and \mathcal{MR}_{red} .*

Proof:

Define $M : \mathcal{ARS}^{op} \rightarrow \mathcal{MR}_{red}$ and $Spec : \mathcal{MR}_{red} \rightarrow \mathcal{ARS}^{op}$ as we already defined in corollary 4.4 and theorem 4.5. Follow by that $M \circ Spec \cong Id_{\mathcal{MR}_{red}}$ and $Spec \circ M \cong Id_{\mathcal{ARS}^{op}}$. ■

5 Special Groups and Multifields

Definition 5.1 (Special Group) A *special group* (SG) is an tuple $(G, -1, \equiv)$, where G is a group of exponent 2, i.e. $g^2 = 1$ for all $g \in G$; -1 is a distinguished element of G , and $\equiv \subseteq G \times G \times G \times G$ is a relation (the special relation), satisfying the following axioms for all $a, b, c, d, x \in G$:

SG 0 - \equiv is an equivalence relation on G^2 ;

SG 1 - $\langle a, b \rangle \equiv \langle b, a \rangle$;

SG 2 - $\langle a, -a \rangle \equiv \langle 1, -1 \rangle$;

SG 3 - $\langle a, b \rangle \equiv \langle c, d \rangle \Rightarrow ab = cd$;

SG 4 - $\langle a, b \rangle \equiv \langle c, d \rangle \Rightarrow \langle a, -c \rangle \equiv \langle -b, d \rangle$;

SG 5 - $\langle a, b \rangle \equiv \langle c, d \rangle \Rightarrow \langle ga, gb \rangle \equiv \langle gc, gd \rangle$, for all $g \in G$.

SG 6 (3-transitivity) - the extension of \equiv for a binary relation on G^3 is a transitive relation.

A group of exponent 2 satisfying SG0-SG5 is called *pre-special group* (PSG). A PSG (or SG) $(G, -1, \equiv)$ is *reduced* (RPSG, RSG respectively) if $1 \neq -1$ and if $\langle a, a \rangle \equiv \langle 1, 1 \rangle \Rightarrow a = 1$.

A *n-form* (or form of dimension $n \geq 1$) is an n -tuple of elements of G . An element $b \in G$ is *represented* on G by the form $\varphi = \langle a_1, \dots, a_n \rangle$, in symbols $b \in D_G(\varphi)$, if there exists $b_2, \dots, b_n \in G$ such that $\langle b, b_2, \dots, b_n \rangle \equiv \varphi$. Now, some examples:

Example 5.2 (The trivial special relation) Let G be a group of exponent 2 and take -1 as any element of G different of 1. For $a, b, c, d \in G$, define $\langle a, b \rangle \equiv_t \langle c, d \rangle$ if and only if $ab = cd$. Then $G_t = (G, \equiv_t, -1)$ is a SG ([3]).

Example 5.3 (Special group of a field) Let F be a field. We denote $F^\bullet = F \setminus \{0\}$, $F^{\bullet 2} = \{x^2 : x \in F^\bullet\}$ and $\Sigma F^{\bullet 2} = \{\sum_{i \in I} x_i^2 : I \text{ is finite and } x_i \in F^{\bullet 2}\}$. Let $G(F) = F^\bullet / F^{\bullet 2}$. In the case of F is be formally real, we have $\Sigma F^{\bullet 2}$ is a subgroup of F^\bullet , then we take $G_{\text{red}}(F) = F^\bullet / \Sigma F^{\bullet 2}$. Note that $G(F)$ and $G_{\text{red}}(F)$ are groups of exponent 2. In [3] they prove that $G(F)$ and $G_{\text{red}}(F)$ are SG's with the special relation given by usual notion of isometry, and $G_{\text{red}}(F)$ is always reduced.

Definition 5.4 A map $(G, \equiv_G, -1) \xrightarrow{f} (H, \equiv_H, -1)$ between PSG's is a *morfism of PSG's* or *PSG-morfism* if $f : G \rightarrow H$ is a homomorfism of groups, $f(-1) = -1$ and for all $a, b, c, d \in G$

$$\langle a, b \rangle \equiv_G \langle c, d \rangle \Leftrightarrow \langle f(a), f(b) \rangle \equiv_H \langle f(c), f(d) \rangle$$

A *morfism of special groups* or *SG-morfism* is a PSG-morfism between the correspondents PSG's. f will be an isomorfism if is bijective and f, f^{-1} are PSG-morfisms.

The category of special groups (respectively reduced special groups) and their morphisms will be denoted by \mathcal{SG} (respectively \mathcal{RSG}). Now, we will analyze the connections between the \mathcal{SG} and \mathcal{MF} . For this, we need more results about special groups and their characterization. For this, we use the results proved in Lira's thesis [1]. Consider these axioms concerns about a group of exponent 2 with a distinguished element:

SG 7 - $\forall a \forall a' \forall x \forall t \forall t' \forall y [(a, a') \equiv (x, t) \wedge (t, t') \equiv (1, y)]$
 $\Rightarrow \exists a'' \exists s \exists s' [(a, a'') \equiv (y, s) \wedge (s, s') \equiv (1, x)]$.

An equivalent statment for SG7 is

$$\bigcup_{t \in D_G(1, y)} D_G(x, t) = \bigcup_{s \in D_G(1, x)} D_G(y, s)$$

for all $x, y \in G$.

SG 8 - For all forms f_1, \dots, f_n of dimension 3 and for all $a, a_2, a_3, b_2, b_3 \in G$,

$$\langle a, a_2, a_3 \rangle \equiv f_1 \equiv \dots \equiv f_n \equiv \langle a, b_2, b_3 \rangle \Rightarrow \langle a_2, a_3 \rangle \equiv \langle b_2, b_3 \rangle.$$

SG 9 - $\forall a \forall b \forall c \forall d [\langle a, b, ab \rangle \equiv \langle c, d, cd \rangle \Rightarrow \langle b, a, ab \rangle \equiv \langle c, d, cd \rangle]$

Proposition 5.5 *Let G be an group of exponent 2 and \equiv_G a binary relation on $G \times G$ verifying the axioms SG0-SG5. Are equivalent:*

i - $\equiv_G \models SG6$

ii - $\equiv_G \models SG7 \wedge SG8$

iii - $\equiv_G \models SG9$

Proof:

[1] page 32. ■

Proposition 5.6 *Let $(G, \equiv, -1)$ be a special group and define $M(G) = G \cup \{0\}$ where $0 := \{G\}^1$. Then $(M(G), +, -, \cdot, 0, 1)$ is a multifield, where*

- $a \cdot b = \begin{cases} 0 & \text{if } a = 0 \text{ or } b = 0 \\ a \cdot b & \text{otherwise} \end{cases}$
- $-(a) = (-1) \cdot a$
- $a + b = \begin{cases} \{b\} & \text{if } a = 0 \\ \{a\} & \text{if } b = 0 \\ M(G) & \text{if } a = -b, \text{ and } a \neq 0 \\ D_G(a, b) & \text{otherwise} \end{cases}$

Proof:

Firstly, observe that $+$ is well-defined. Then, we will verify the conditions of definition 1.5:

¹Here, the choice of the zero element was ad hoc. Indeed, we can define $0 := \{x\}$ for any $x \notin G$.

i - For this, we will check the conditions of definition 1.1.

- a - $d \in a + 0 = \{a\}$ imply $d = a$, and by this, follow that $a \in d + (-0)$ and $0 \in (-a) + d$. Let $a = -b$ and $d \in a + (-a) = M(G)$. If $d = 0$, then $a \in d + (-(-a)) = 0 + a$ and $-a \in (-a) + 0$. If $d \neq 0$, then $a \in D_G(d, a)$ and $-a \in D_G(-a, d)$ so $a \in d + (-(-a)) = d + a$ and $-a \in (-a) + d$. Finally, let $a, b \neq 0$ with $a \neq -b$, and $d \in a + b$. Then there exist $g \in M(G) \setminus \{0\}$ such that $\langle d, g \rangle \equiv \langle a, b \rangle$. By SG4, $\langle d, -a \rangle \equiv \langle -g, b \rangle$ (and $\langle b, -g \rangle \equiv \langle -a, d \rangle$ by SG1). So $a \in d + (-b)$ and $b \in (-a) + d$.
- b - $(y \in x + 0) \Leftrightarrow (x = y)$ is direct consequence of the definition of sum.
- c - $a + 0 = 0 + a$ and $a + (-a) = M(G) = (-a) + a$. Let $a, b \in M(G)$, $a, b \neq 0$ and $a \neq -b$. How $D_G(a, b) = D_G(b, a)$, we have $a + b = b + a$. Then, the commutativity holds. Observe that if $a, b \neq 0$ with $a \neq -b$, then $0 \notin a + b$.
- d - Now we prove the associativity. Let $a = 0$ (the cases $b = 0$ and $c = 0$ are analogous). Then $0 + (b + c) = \{0 + g : g \in b + c\} = b + c$ and $(0 + b) + c = (\{b\}) + c = b + c$.
Now, let $a, b, c \neq 0$ with $a = -c$.

$$(a + b) + (-a) = \bigcup \{g + (-a) : g \in a + b\} = M(G) \text{ (I)}$$

because $a \in a + b$, and

$$a + (b + (-a)) = \bigcup \{a + h : h \in b + (-a)\} = M(G) \text{ (II)}$$

because $-a \in b + (-a)$. So (I) = (II) and $(a + b) + (-a) = a + (b + (-a))$. For the case $a, b, c \neq 0$, $a = -b$ (the cases $b \neq -c$ is analogous) we have

$$(a + (-a)) + c = \bigcup \{g + c : g \in M(G)\} = M(G) \text{ (III)}$$

and

$$a + ((-a) + c) = \bigcup \{a + h : h \in (-a) + c\} = M(G) \text{ (IV)}$$

because $-a \in (-a) + c$. So (III) = (IV) and $(a + (-a)) + c = a + ((-a) + c)$. Finally, let $a, b, c \neq 0$, $a \neq -b$, $b \neq -c$ and $a \neq -c$.

$$(a + b) + c = c + (a + b) = \bigcup \{c + g : g \in a + b\} = \bigcup_{g \in D_G(a, b)} D_G(c, g) \text{ (V)}$$

and

$$a + (b + c) = \bigcup \{h + a : h \in b + c\} = \bigcup_{h \in D_G(b, c)} D_G(h, a) \text{ (VI)}$$

By SG7 (applying SG5) we have (V) = (VI). Then $(a + b) + c = a + (b + c)$ for all $a, b, c \in M(G)$.

- ii - We conclude that $(M(G), \cdot, 1)$ is a commutative monoid as consequence of $(G, \cdot, 1)$ is an abelian group and the extended definition of \cdot to $M(G)$. Beyond this, we have that every nonzero element of $M(G)$ has an inverse.

- iii - $a \cdot 0 = 0$ for all $a \in M(G)$ is a consequence of the extended definition of multiplication to $M(G)$.
- iv - If $a = 0$ or $a \neq -b$, then $(d \in a + b) \Rightarrow \forall g(gd \in ga + gb)$ is direct consequence of the definition of sum. Next this, let $a, b \neq 0$ with $a \neq -b$ and $d \in a + b$. By SG5 $gd \in ga + bg$. Thus we have $g(a + b) \subseteq ga + gb$ for all $a, b, g \in M(G)$.

Then, $(M(G), +, -, \cdot, 0, 1)$ is a multifield. ■

Corollary 5.7 *The correspondence $G \mapsto M(G)$ defines a faithful functor $M : \mathcal{SG} \rightarrow \mathcal{MF}$.*

Proof:

Let $f : G \rightarrow H$ be a SG-morphism. We will extend f to $M(f) : M(G) \rightarrow M(H)$ by $M(f) \upharpoonright_G = f$ and $M(f)(0) = 0$. By the definition of SG-morphism we have $M(f)(1) = 1$, $M(f)(-a) = -a$ and $M(f)(ab) = M(f)(a)M(f)(b)$. As $d \in D_G(a, b)$ implies $f(d) \in D_H(f(a), f(b))$ we have $d \in a + b \Rightarrow M(f)(d) \in M(f)(a) + M(f)(b)$ for all $a, b \in M(G)$. So $M(f)$ is a multiring morphism.

Now, let $G \xrightarrow{f} H \xrightarrow{g} K$ be SG-morphisms. How $M(f \circ g) \upharpoonright_G = f \circ g = M(f) \upharpoonright_G \circ M(g) \upharpoonright_G$ and $M(f \circ g)(0) = 0 = M(f) \circ M(g)(0)$, we have $M(f \circ g) = M(f) \circ M(g)$. Then $M : \mathcal{SG} \rightarrow \mathcal{MF}$ is a functor.

This functor is faithful, because if G and H are special groups and $f, g : G \rightarrow H$ are SG-morphisms such that $M(f), M(g) : M(G) \rightarrow M(H)$ are equal, then $M(f)|_{M(G) \setminus \{0\}} = M(g)|_{M(G) \setminus \{0\}}$ and therefore $f = g$, since $M(G) \setminus \{0\} = G$. ■

Proposition 5.8 *Let G be an SG and $M(G)$ as above. Then:*

- i - $a^2 = 1$ for all $a \in M(G)^\bullet$;
- ii - $a + (-a) = 0$ for all $a \in M(G)^\bullet$;
- iii - If $ab = cd$ and $a \in c + d$ then $c \in a + b$ for all $a, b, c, d \in M(G)^\bullet$;
- iv - If $ab = cd = ef$, $a \in c + d$ and $c \in e + f$ then $a \in e + f$ for all $a, b, c, d, e, f \in M(G)^\bullet$;
- v - If there exists $x, y, z \in F^\bullet$ such that

$$\begin{cases} ax = cy \\ a = xz \\ c = yz \end{cases} \quad \text{and} \quad \begin{cases} a \in c + y \\ b \in x + z \\ d \in y + z \end{cases}$$

then there exists $t, v, w \in F^\bullet$ such that

$$\begin{cases} bt = cv \\ b = tw \\ d = vw \end{cases} \quad \text{and} \quad \begin{cases} b \in c + v \\ a \in t + w \\ d \in v + w \end{cases}$$

Proof:

- i - Is just the fact of G be a group of exponent 2.
- ii - Is the extended definition os sum.
- iii - Is the symmetry of \equiv_G .
- iv - Is the transitivity of \equiv_G .
- v - Is the 3-transitivity.

■

Definition 5.9 A multifield F satisfying the properties I-V of proposition 5.8 will be called *aspecial multifield* (SMF). Note that, if G is a SG, then $M(G)$ is a SMF.

Theorem 5.10 If F is a special multifield the $(F^\bullet, \equiv, -1)$ is a special group where $\langle a, b \rangle \equiv \langle c, d \rangle \Leftrightarrow ab = cd$ and $a \in c + d$.

Proof:

By (i), we have that $(F^\bullet, 1)$ is a group of exponent 2. Now, we will check each axiom of definition 5.1:

SG0 - By (ii) $1 \in ab - ab$, so $ab \in 1 + ab$ and $a \in b + a$. As $ab = ab$, then $\langle a, b \rangle \equiv \langle a, b \rangle$, i.e., \equiv is reflexive. If $\langle a, b \rangle \equiv \langle c, d \rangle$, then $ab = cd$ and $a \in c + d$. By (iii), $c \in a + b$, so $\langle c, d \rangle \equiv \langle a, b \rangle$ and \equiv is symmetric. Finally, suppose that $\langle a, b \rangle \equiv \langle c, d \rangle$ and $\langle c, d \rangle \equiv \langle e, f \rangle$. First, $ab = cd$ and $cd = ef$ implies $ab = ef$. Second, $a \in c + d$ and $c \in e + f$ by (iv), we have $a \in e + f$. Therefore $\langle a, b \rangle \equiv \langle e, f \rangle$.

SG1 - As F is a multifield, $ab = ba$. By (ii), $1 \in ab - ba$, then $ab \in 1 + ba$ and $b \in a + b$. Therefore $\langle a, b \rangle \equiv \langle b, a \rangle$.

SG2 - $a \cdot 1 = (-a) \cdot (-1)$ and by (ii), $1 \in a + (-a)$, then $a = a \cdot 1 \in 1 + a = 1 + (-a) \cdot (-1)$. Therefore $\langle a, -a \rangle \equiv \langle 1, -1 \rangle$.

SG3 - Follow by definition.

SG4 - $\langle a, b \rangle \equiv \langle c, d \rangle \Rightarrow ab = cd$ and $a \in c + d$.

$$ab = cd \Rightarrow -abbc = -bccd \Rightarrow -ac = -bd \quad (3)$$

$$\begin{aligned} a \in c + d &\Rightarrow c \in a + acd \Rightarrow c \in a + b \Rightarrow b \in a - c \Rightarrow \\ b \in -a + c &\Rightarrow -ab \in 1 - ac \Rightarrow -ab \in 1 - bd \Rightarrow a \in -b + d \end{aligned} \quad (4)$$

so by 3 and 4 follow that $\langle a, -c \rangle \equiv \langle -b, d \rangle$.

SG5 - $\langle a, b \rangle \equiv \langle c, d \rangle \Rightarrow ab = cd$ and $a \in c + d \xrightarrow{I} (ga)(gb) = (gc)(gd)$ and $ga \in gc + gd \Rightarrow \langle ga, gb \rangle \equiv \langle gc, gd \rangle$.

SG9 - $\langle a, b, ab \rangle \equiv \langle c, d, cd \rangle \Rightarrow$ there exists $x, y, t \in F^\bullet$ such that

$$\begin{cases} \langle a, x \rangle \equiv \langle c, y \rangle \\ \langle b, ab \rangle \equiv \langle x, z \rangle \\ \langle d, cd \rangle \equiv \langle y, z \rangle \end{cases} \Rightarrow \begin{cases} ax = cy \text{ and } a \in c + y \\ a = xz \text{ and } b \in x + z \\ c = yz \text{ and } d \in y + z \end{cases}$$

then by (v) there exists $t, v, w \in F^\bullet$ such that

$$\begin{cases} bt = cv \text{ and } b \in c + v \\ b = tw \text{ and } a \in t + w \\ d = vw \text{ and } d \in v + w \end{cases} \Rightarrow \begin{cases} \langle b, t \rangle \equiv \langle c, v \rangle \\ \langle a, ab \rangle \equiv \langle t, w \rangle \\ \langle d, cd \rangle \equiv \langle v, w \rangle \end{cases}$$

this implies $\langle b, a, ab \rangle \equiv \langle c, d, cd \rangle$. ■

Corollary 5.11 *There is a functor $S : \mathcal{SMF} \rightarrow \mathcal{SG}$.*

Proof:

In the objects of \mathcal{SMF} , we define $S(F) = F^\bullet$ how the special group as stated in theorem 5.10. Now, let $\sigma : F \rightarrow K$ be a SMF-morphism. Define $S(\sigma) = \sigma|_{F^\bullet}$. We have that $S(\sigma)$ is a group homomorphism with $S(\sigma)(-1) = -1$. If $a, b \neq 0$ and $c \in a + b$, $c \neq 0$, then there exists $d \in F^\bullet$ such that $\langle a, b \rangle \equiv_{S(F)} \langle c, d \rangle$, and as $c \in a + b \rightarrow \sigma(c) \in \sigma(a) + \sigma(b)$, we have $\langle \sigma(a), \sigma(b) \rangle \equiv_{S(K)} \langle \sigma(c), \sigma(d) \rangle$. Therefore:

$$(c \in a + b \rightarrow \sigma(c) \in \sigma(a) + \sigma(b)) \Rightarrow (c \in D_{S(F)}(a, b) \rightarrow \sigma(c) \in D_{S(K)}(\sigma(a), \sigma(b)))$$

And $S(\sigma)$ is a SG-morphism. Applying the same argument, we proof that $S(\sigma\tau) = S(\sigma)S(\tau)$. Hence, S is a morphism. ■

Theorem 5.12 *There exist an equivalence of categories between \mathcal{SG} and \mathcal{SMF} .*

Proof:

By the corollaries 5.7 and 5.11, we have functors $M : \mathcal{SG} \rightarrow \mathcal{SMF}$ and $S : \mathcal{SMF} \rightarrow \mathcal{SG}$. We will proof that $M \circ S \cong Id_{\mathcal{SMF}}$ and $S \circ M \cong Id_{\mathcal{SG}}$.

- i - $M \circ S \cong Id_{\mathcal{SMF}}$. Let F be a SMF. How $S(F) = F^\bullet$ and $M(S(F)) = S(F) \cup \{0\}$, we have $M(S(F)) = F$. Next, let $\sigma : F \rightarrow K$ be a SMF-morphism. We have that $S(\sigma) = \sigma|_{F^\bullet}$ and $M(S(\sigma))$ is defined with the extension $S(\sigma)(0) = 0$. Therefore $M(S(\sigma)) = \sigma$ and $M \circ S \cong Id_{\mathcal{SMF}}$.
- ii - $S \circ M \cong Id_{\mathcal{SG}}$. Let G be a SG. Again, $M(G) = G \cup \{0\}$ and $S(M(G)) = M(G) \setminus \{0\}$. Hence $S(M(G)) = G$. Next, let $f : G \rightarrow H$ be a SG-morphism. How $M(f)$ is defined with the extension $f(0) = 0$ and $S(M(f)) = M(f)|_{M(G) \setminus \{0\}}$, we have that $S(M(f)) = f$ and $S \circ M \cong Id_{\mathcal{SG}}$, finalizing the proof. ■

6 Real semigroups, Reduced Special Groups and Multirings

Definition 6.1 (Ternary Semigroup) A *ternary semigroup* (abbreviated TS) is a structure $(S, \cdot, 1, 0, -1)$ with individual constants $1, 0, -1$ and a binary operation “ \cdot ” such that:

TS1 - $(S, \cdot, 1)$ is a commutative semigroup with unity;

TS2 - $x^3 = x$ for all $x \in S$;

TS3 - $-1 \neq 1$ and $(-1)(-1) = 1$;

TS4 - $x \cdot 0 = 0$ for all $x \in S$;

TS5 - For all $x \in S$, $x = -1 \cdot x \Rightarrow x = 0$.

We shall write $-x$ for $(-1) \cdot x$. The semigroup verifying conditions [TS1] and [TS2] (no extra constants) will be called *3-semigroups*.

Example 6.2

a - The three-element structure $\mathbf{3} = \{1, 0, -1\}$ has an obvious ternary semigroup structure.

Here, we will enrich the language $\{\cdot, 1, 0, -1\}$ with a ternary relation D . In agreement with 4.1, we shall write $a \in D(b, c)$ instead of $D(a, b, c)$. We also set:

$$a \in D^t(b, c) \Leftrightarrow a \in D(b, c) \wedge -b \in D(-a, c) \wedge -c \in D(b, -a).$$

The relations D and D^t are called *representation* and *transversal representation* respectively.

Definition 6.3 (Real Semigroup) A *real semigroup* (abbreviated RS) is a ternary semigroup together with a ternary relation D satisfying:

RS0 - $c \in D(a, b)$ if and only if $c \in D(b, a)$.

RS1 - $a \in D(a, b)$.

RS2 - $a \in D(b, c)$ implies $ad \in D(bd, cd)$.

RS3 (Strong Associativity) - If $a \in D^t(b, c)$ and $c \in D^t(d, e)$, then there exists $x \in D^t(b, d)$ such that $a \in D^t(x, e)$.

RS4 - $e \in D(c^2a, d^2b)$ implies $e \in D(a, b)$.

RS5 - If $ad = bd$, $ae = be$ and $c \in D(d, e)$, then $ac = bc$.

RS6 - $c \in D(a, b)$ implies $c \in D^t(c^2a, c^2b)$.

RS7 (Reduction) - $D^t(a, -b) \cap D^t(b, -a) \neq \emptyset$ implies $a = b$.

RS8 - $a \in D(b, c)$ implies $a^2 \in D(b^2, c^2)$.

The definition of morphism is quite standard: $f : (G, \cdot, 1, 0-1) \rightarrow (H, \cdot, 1, 0-1)$ is an *RS-morphism* if $f : G \rightarrow H$ is a morphism of semigroups, (i.e, $f(ab) = f(a)f(b)$, $f(1) = 1$ and $f(0) = 0$); $f(-1) = -1$ and $a \in D(b, c) \Rightarrow f(a) \in D(f(b), f(c))$ (hence $a \in D^t(b, c) \Rightarrow f(a) \in D^t(f(b), f(c))$). The category of realsemigroups and their morphisms will be denoted by \mathcal{RS} .

Proposition 6.4 *The properties below holds in any realsemigroup G , for arbitrary $a, b, c, d, e \in G$:*

$$i - a \in D^t(b, c) \Rightarrow -b \in D^t(-a, c).$$

$$ii - 0 \in D(a, b).$$

$$iii - a \in D^t(b, c) \Rightarrow ad \in D^t(bd, cd).$$

$$iv - a \in D(0, 1) \cup D(1, 1) \Rightarrow a = a^2.$$

$$v - d \in D(ca, cb) \Rightarrow d = c^2d.$$

$$vi - a^2 \in D(1, b). \text{ Hence (by (iv)), } Id(G) = D(1, 1).$$

$$vii - a \in D^t(b, b) \Leftrightarrow a = b.$$

$$viii - a \in D(0, 0) \Leftrightarrow a = 0.$$

$$ix - 1 \in D^t(1, a).$$

$$x - D^t(1, -1) = G.$$

$$xi - ab \in D(1, -a^2).$$

$$xii - 0 \in D^t(a, b) \Leftrightarrow a = -b.$$

$$xiii - a \in D(b, c) \wedge b, c \in D(x, y) \Rightarrow a \in D(x, y).$$

$$xiv - a \in D(b, c) \Leftrightarrow ab \in D(1, bc) \wedge ac \in D(1, bc) \wedge a^2 \in D(b^2, c^2).$$

$$xv - D^t(a, b) \neq \emptyset.$$

$$xvi - (\text{Weak Associativity}) \ a \in D(b, c) \wedge c \in D(d, e) \Rightarrow \exists x[x \in D(b, d) \wedge a \in D(x, e)].$$

$$xvii - a \in D(b, c) \Leftrightarrow a \in D^t(a^2b, a^2c).$$

Proof:

The itens (i)-(xvi) is just proposition 2.3 in [2]. For the item (xvii), (\Rightarrow) is just RS6 and (\Leftarrow) is RS4 (or RS2 + TS2). \blacksquare

Proposition 6.5 *The ternary semigroup $\mathbf{3} = \{1, 0, -1\}$ has a unique structure of real semigroup, with representation given by:*

$$\begin{cases} D_{\mathbf{3}}(0, 0) = \{0\}; \\ D_{\mathbf{3}}(0, 1) = D_{\mathbf{3}}(1, 0) = D_{\mathbf{3}}(1, 1) = \{0, 1\}; \\ D_{\mathbf{3}}(0, -1) = D_{\mathbf{3}}(-1, 0) = D_{\mathbf{3}}(-1, -1) = \{0, -1\}; \\ D_{\mathbf{3}}(1, -1) = D_{\mathbf{3}}(-1, 1) = \mathbf{3} \end{cases}$$

and transversal representation given by:

$$\begin{cases} D_{\mathbf{3}}^t(0, 0) = \{0\}; \\ D_{\mathbf{3}}^t(0, 1) = D_{\mathbf{3}}^t(1, 0) = D_{\mathbf{3}}^t(1, 1) = \{1\}; \\ D_{\mathbf{3}}^t(0, -1) = D_{\mathbf{3}}^t(-1, 0) = D_{\mathbf{3}}^t(-1, -1) = \{-1\}; \\ D_{\mathbf{3}}^t(1, -1) = D_{\mathbf{3}}^t(-1, 1) = \mathbf{3} \end{cases}$$

Proof:

See corollary 2.4 in [2]. ■

Theorem 6.6 (Separation Theorem from [2]) *Let G be a RS, and $a, b, c \in G$ and $X_G = \text{Hom}(G, \mathbf{3})$. Then:*

- i - $a \in D_G(b, c)$ if and only if for all $h \in X_G$, $h(a) \in D_{\mathbf{3}}(h(b), h(c))$.*
- ii - $a \in D_G^t(b, c)$ if and only if for all $h \in X_G$, $h(a) \in D_{\mathbf{3}}^t(h(b), h(c))$.*
- iii - If $a \neq b$, there is $h \in X_G$ such that $h(a) \neq h(b)$.*

Proof:

See theorem 4.4 in [2]. ■

Theorem 6.7 *Let $(G, \cdot, 1, 0, -1, D)$ be a realsemigroup and define $+: G \times G \rightarrow \mathcal{P}(G) \setminus \{\emptyset\}$, $a + b = \{d \in G : d \in D^t(a, b)\}$ and $-: G \rightarrow G$ by $-(g) = -1 \cdot g$. Then $(G, +, \cdot, -, 0, 1)$ is a real reduced multiring.*

Proof:

Firstly, observe that by 6.4(xv) the sum is well-defined, i.e, $D^t(a, b) \neq \emptyset$ for all $a, b \in G$.

Now, we will check that G is a multiring: of course, by RS0 we have $a + b = b + a$ (i.e, $D^t(a, b) = D^t(b, a)$) and

$$\begin{cases} d \in D^t(a, b) \Leftrightarrow d \in D(a, b) \wedge -a \in D(-d, b) \wedge -b \in D(a, -d) \\ a \in D^t(d, -b) \Leftrightarrow a \in D(d, -b) \wedge -d \in D(-a, -b) \wedge b \in D(d, -a) \\ b \in D^t(-a, d) \Leftrightarrow b \in D(-a, d) \wedge a \in D(-b, d) \wedge -d \in D(-a, -b) \end{cases}$$

So $d \in D^t(a, b) \Rightarrow a \in D^t(d, -b) \wedge b \in D^t(-a, d)$, or in other words, $d \in a + b \Rightarrow a \in d + (-b) \wedge b \in (-a) + d$. If $x = y$, by RS1 $x \in 0 + y$. Conversely, let $x \in 0 + y$. We just proved that $0 \in x - y$ and $0 \in y - x$ then by RS7, $x = y$. How RS3 states the associativity (like 1.3) we have that G is a commutative multigroup.

Because the commutative semigroup structure of $(G, \cdot, -1, 0, 1)$, we have that $(G, \cdot, 1)$ is a commutative monoid and $a \cdot 0 = 0$ for all $a \in G$. The distributive law is just 6.4(iii), we have that G is a multiring.

Finally, we prove that G is real reduced. We already have that $-1 \neq 0$ and $a^3 = a$. We have too, that $1 \in D^t(1, b^2)$ by 6.4(ix) then by 6.4(iii) $a \in D^t(a, ab^2)$. Now, how $t^3 = t$ we have

$$\begin{aligned} t \in D^t(v^2x, w^2y) &\Leftrightarrow t \in D(v^2x, w^2y) \wedge -v^2x \in D(-t^3, w^2y) \wedge -w^2y \in D(v^2x, -t^3) \\ &\stackrel{\text{RS4}}{\Leftrightarrow} t \in D(x, y) \wedge -v^2x \in D(-t, y) \wedge -w^2y \in D(x, -t) \end{aligned} \tag{5}$$

Hence, how by RS1 $-a \in D(-a, -x)$ for all $a, x \in G$, follow

$$\begin{aligned} x \in D^t(a, ab^2) &\Leftrightarrow x \in D^t(a^2 \cdot a, (ab)^2 \cdot a) \stackrel{5}{\Leftrightarrow} \\ x \in D(a, a) \wedge -a \in D(-x, a) \wedge -ab^2 \in D(a, -x) &\Leftrightarrow \\ [x \in D(a, a) \wedge -a \in D(-x, a) \wedge -a \in D(a, -x)] \wedge -ab^2 \in D(a, -x) &\Leftrightarrow \\ x \in D^t(a, a) \wedge -ab^2 \in D(a, -x) &\stackrel{6.4(vii+x)}{\Leftrightarrow} x = a \end{aligned}$$

Then $a + ab^2 = \{a\}$. For the last property, we have by theorem 6.6(ii), we have that $d \in D^t(b^2, c^2) \Leftrightarrow h(d) \in D_{\mathbf{3}}^t(h(b^2), h(c^2))$ for every $h \in X_G$. How by proposition 6.5 $D^t(t^2, s^2)$ is unitary for every $s, t \in \mathbf{3}$, we have that $D^t(b^2, c^2)$ is unitary for every $b, c \in G$.

Hence, by definition 2.9 G is a real reduced multiring. \blacksquare

Corollary 6.8 *There is a functor $M : \mathcal{RS} \rightarrow \mathcal{MR}_{red}$.*

Proof:

Let $R, S \in \mathcal{RS}$ and $f : R \rightarrow S$ a RS-morphism. Define $M(R)$ how the real reduced multiring as in theorem 6.7 and $M(f) = f$. Of course, $M(f)$ is a multiring morphism, because $c \in a + b \Rightarrow c \in D^t(a, b) \Rightarrow f(c) \in D^t(f(a), f(b)) \Rightarrow f(c) \in f(a) + f(b)$. This is suffice to prove that M is a functor. \blacksquare

Lemma 6.9 $(Q_2, \cdot, 1, 0, -1, D)$ (example 1.6(b)) is a realsemigroup, when $D^t(a, b) = a + b$ and $d \in D(a, b) \Leftrightarrow d \in d^2a + d^2b$.

Proof:

Remember that $Q_2 = \{-1, 0, 1\}$ is the multifold when \cdot the usual product and the sum is defined by

$$\begin{cases} 0 + x = x + 0 = x, \text{ for every } x \in Q_2 \\ 1 + 1 = 1, (-1) + (-1) = -1 \\ 1 + (-1) = (-1) + 1 = \{-1, 0, 1\} \end{cases}$$

We also have that:

$$a \in D^t(b, c) \Leftrightarrow a \in a^2b + a^2c \wedge -b \in -b^2a + b^2c \wedge -c \in c^2b - c^2a,$$

because $x^2 \in \{0, 1\}$ for all $x \in Q_2$. Now, we will check each axiom of definition 6.3:

RS0 - Is just commutativity of sum.

RS1 - Analyzing each case, we have $a \in a + a^2b = a^3 + a^2b$ for every $a, b \in Q_2$, then $a \in D(a, b)$.

RS2 - $a \in D(b, c) \Leftrightarrow a \in a^2b + a^2c \stackrel{d^3=d}{\Rightarrow} ad \in (ad)^2bd + (ad)^2cd \Rightarrow ad \in D(bd, cd)$.

RS3 - Is just associativity of sum.

RS4 - If $c \neq 0$ and $d \neq 0$, then $c^2 = d^2 = 1$, and $D(c^2a, d^2b) = D(a, b)$. If $c = 0$, we have $D(c^2a, d^2b) = D(0, d^2b)$, and $e \in D(0, d^2b) \Leftrightarrow e \in 0 + d^2b$ then $e = d^2b$. If $d = 0$, then $e = 0$ and $e \in D(0, b)$. If $d \neq 0$, $d^2 = 1$ and $e = b$, and $e \in D(0, b)$.

RS5 - Is just the compute of each case, in the same line of RS4.

RS6 - Is just the definition of D .

RS7 - Follow by item (ii) of definition 1.1, i.e, $y \in 0 + x \Leftrightarrow x = y$.

RS8 - Is just the compute of each case, in the same line of RS4.

Then, $(Q_2, \cdot, 1, 0, -1, D)$ is a realsemigroup. ■

Theorem 6.10 *Let A be a real reduced multiring. Then $(A, \cdot, 1, 0, -1, D)$ is a realsemigroup, when $D^t(a, b) = a + b$ and $d \in D(a, b) \Leftrightarrow d \in d^2a + d^2b$.*

Proof:

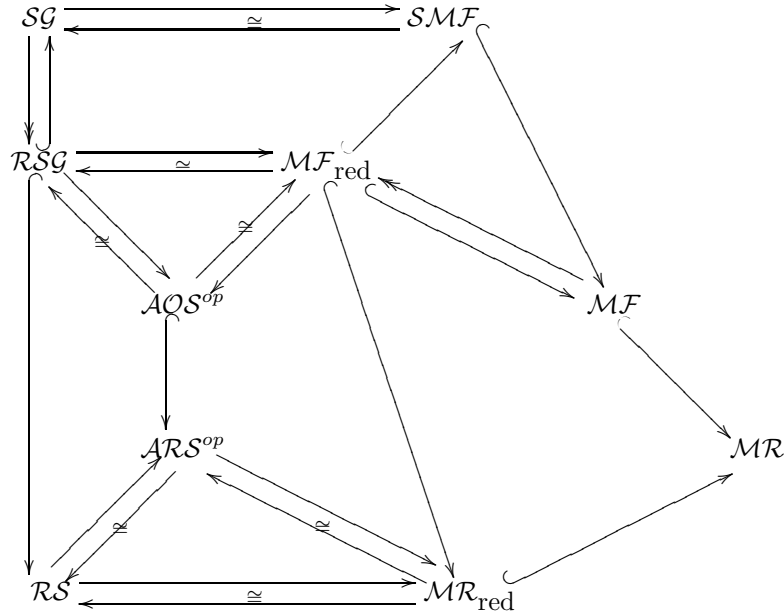
By the local-global principle for multirings, is suffice prove the case $A = Q_2$, and this case is the lemma 6.9 above. ■

Corollary 6.11 *There exist an equivalence of categories between \mathcal{RS} and \mathcal{MR}_{red} .*

Proof:

Define the functor $S : \mathcal{MR}_{red} \rightarrow \mathcal{RS}$ as in corollary 6.8. The proof of $S \circ M \cong Id_{\mathcal{RS}}$ and $M \circ S \cong Id_{\mathcal{MR}_{red}}$ is mutatis mutandis of theorem 5.12. ■

Finally, we provide a diagram for a better visualization of the functors obtained:



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